Chapter 2

Fuzzy Relations

Fuzzy Relation R is a mapping from the Cartesian space A × B to interval [0, 1] where the strength of mapping is expressed by the membership function of the relation.

$$R \subseteq A \times B \quad \text{And} \quad \mu_R = A \times B \rightarrow [0,1]$$

Example:

A simple example of a binary fuzzy relation on U = {1, 2, 3}, called "approximately equal" can be defined as

R(1, 1) = R(2, 2) = R(3,3) = 1
R(1, 2) = R(2, 1) = R(2, 3) = R(3,2) = 0.8
R(1, 3) = R(3,1) = 0.3

In other words, the membership function of R is given by

$$R(u, v) = \begin{cases} 1 & \text{if } u = v \\ 0.8 & \text{if } |u - v| = 1 \\ 0.3 & \text{if } |u - v| = 2 \end{cases}$$

In matrix notation it can be represented as

R =

	1	2	3
1	1.0	0.8	0.3
2	0.8	1.0	0.8
3	0.3	0.8	1.0

Graphical representation of a fuzzy relation: A relation R also can be graphically represented. For example, suppose, a fuzzy relation R in V × W, where V = {1,2,3) and W = {1,2,3,4} has the following definition.

R = [{{{1,1},1},{{1,2},.2},{{1,3},.7},{{1,4},0},
{{2,1},.7},{{2,2},1},{{2,3},.4},{{2,4},.8},
{{3,1},0},{{3,2},.6},{{3,3},.3},{{3,4},.5},

This relation can be represented in the form of a graph as shown in Fig.1.

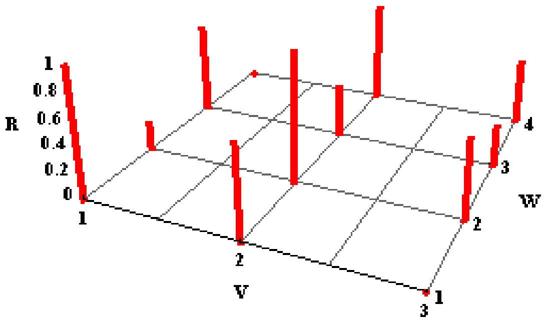


Figure 1: Graphical representation of a relation R

Operations on Fuzzy Relations

Let R and S be two fuzzy relations on Cartesian space X × Y. There are three major Fuzzy operations:

1. Union

The union of R and S is defined by (R ∨ S)(u, v) = max{R(u, v), S(u, v)}

As we know union of sets is to combine all the elements of those sets involved in that operation, in cas of fuzzy relations(which are also sets), we not only combine their elements but also calculate their membership function as:

$$\mu_{R \cup S}(x, y) = Max\,[\,\mu_R(x, y), \mu_S(y, z)\,]$$

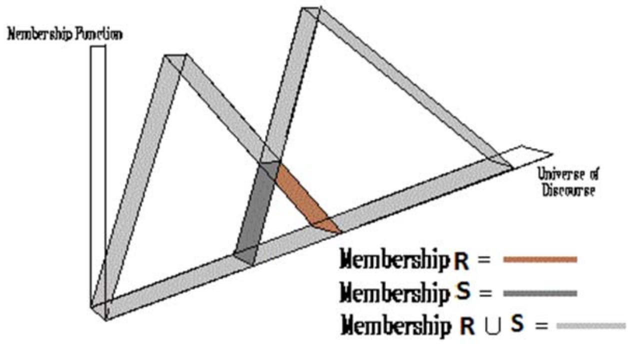


Figure 2: Graphical representation of union operation of two relations R and S

Example:

Suppose, two relations R and S are given as follows.

R = "x is considerable larger than y"

	y1	y2	y3	y4
x1	0.8	0.1	0.1	0.7
x2	0.0	0.8	0.0	0.0
x3	0.9	1.0	0.7	0.8

S = "x is very close to y"

	y1	y2	y3	y4
x1	0.4	0.0	0.9	0.6
x2	0.9	0.4	0.5	0.7
x3	0.3	0.0	0.8	0.5

The union of R and S means that "x is considerable larger than y" **or** "x is very close to y".

(R ∨ S)(x, y) =

	y1	y2	y3	y4
x1	0.8	0.0	0.9	0.7
x2	0.9	0.8	0.5	0.7
x3	0.1	0.0	0.3	0.2

2. Intersection

The intersection of R and S is defined as

(R ∧ S)(u, v) = min{R(u, v), S(u, v)}.

Similarly in case intersection of fuzzy relations we find the membership function as:

$$\mu_{R \cap S} = Min\, [\, \mu_R\,(x, y), \mu_S\,(y, z)\,]$$

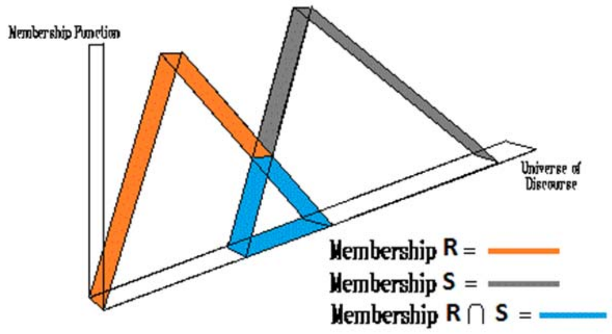


Figure 3: Graphical representation of intersection operation of two relations R and S

With reference to the relations R and S as mentioned above, the intersection means that "x is considerable larger than y" **and** "x is very close to y".

(R ∧ S)(x, y) =

	y1	y2	y3	y4
x1	0.4	0.0	0.1	0.6
x2	0.0	0.4	0.0	0.0
x3	0.3	0.0	0.7	0.5

3. Complement

The Complement of R is defined R and can be stated as $\bar{R}(u,v) = 1 - R(u,v)$.

Here to make a complement of a fuzzy relation, we add all the elements of the primary relation and just make the membership function change as follows:

$$\mu_{R^c}(x, y) = 1 - \mu_R(x, y)$$

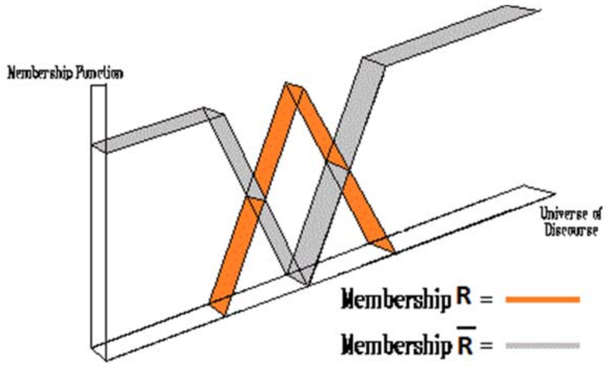


Figure 4: Graphical representation of intersection operation of two relations R and S

The Complement of R means that "x is considerably smaller than y".

R̄ =

	y1	y2	y3	y4
x1	0.2	0.9	0.9	0.3
x2	1.0	0.2	1.0	1.0
x3	0.9	1.0	0.8	0.8

In addition to the above mentioned operations on relations, there are few more operations, which are stated below.

4. Containment

This operation is to test whether a fuzzy relation is a complete subset of other that is we say R contains in S if and only if the following condition holds:

$$R \subset S \equiv \mu_R(x, y) \leq \mu_S(x, y)$$

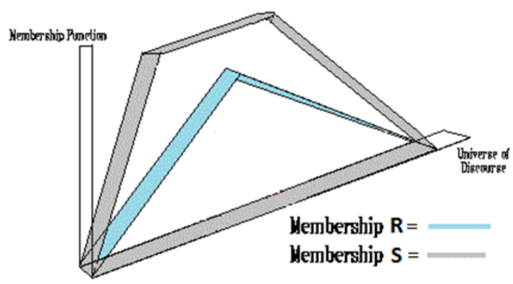


Figure 5: Graphical representation of intersection operation of two relations R and S

5. Projection

The projection of R on X is defined as follows:

$\Pi_X(x) = \max\{R(x,y) | y \in Y\}$

And the projection of R on Y is defined as

$\Pi_Y(y) = \max\{R(x,y) | x \in X\}$

Example:

$$R = \begin{bmatrix} x/y & y1 & y2 & y3 & y4 \\ x1 & .3 & .8 & .7 & .5 \\ x2 & .7 & .3 & .5 & .4 \\ x3 & .9 & 0 & .2 & .3 \end{bmatrix}$$

$\Pi_X(x1) = \max(.3, .8, .7, .5) = .8$

$\Pi_Y(y2) = \max \begin{pmatrix} .8 \\ .3 \\ 0 \end{pmatrix} = .8$

NOW,

Let R and S be two fuzzy relations on Cartesian space X × Y and Y × Z respectively. Then the membership function of the operation composition on these two fuzzy relations is as follows:

6. Composition

The composition operation combines fuzzy relations over different product spaces. We here discuss two import operations max-min composition and max-product composition which are very popular.

This operation is basically gives us the flexibility to combine two or more relations on fuzzy relations, this can be defined in many ways as suitable; here is one general definition which is also known as min-max operation. The definition is a s follows:

$$R \circ S = \{ (x,z) \mid (x,y) \in R_{X \times Y}, (y,z) \in S_{Y \times Z} \}$$

$$\mu_{R \circ S}(x,y) = \max_{\forall y \in Y} \{min\{ \mu_R(x,y), \mu_S(y,z)\}\} : R \in X \times Y, S \in Y \times Z$$

Example 1:

Let's take the following fruit example with the colour-maturity relation R

R	Verdant	Half-mature	Mature
Green	1	0.2	0
Yellow	0.3	1	0.4
Red	0	0.2	1

And also define for a maturity-taste relation S

S	Sour	Tasteless	Sweet
Verdant	1	0.2	0
Half-mature	0.7	1	0.3
Mature	0	0.7	1

Then by applying the composition operation to the elements of these two tables, the following is obtained:

$T \equiv R \circ S$	Sour	Tasteless	Sweet
Green	1	0.2	0
Yellow	0.7	1	0.3
Red	0	0.7	1

Example 2

Two fuzzy relation $R = [r_{ij}]$ and $S = [s_{jk}]$ expressed in matrix form, on sets A, B and C i.e. $R \subseteq A \times B$ and $S \subseteq B \times C$. The Composition T of two relations R and S is expressed as $T = R \circ S$ is a fuzzy relation, whose membership function is related to those of R and S as follows:

$$[t_{ik}] = [r_{ij}] \circ [s_{jk}] \text{ Where } t_{ik} = \max[\min(r_{ij}, s_{jk})]$$

Assume the Min-Max Composition.

Consider two relations R and S as follows:

R = "x is considerable larger than y"

	y1	y2	y3	y4
x1	0.8	0.1	0.1	0.7
x2	0.0	0.8	0.0	0.0
x3	0.9	1.0	0.7	0.8

S = "y is very close to z"

	z1	z2	z3
y1	0.4	0.9	0.3
y2	0.0	0.4	0.0
y3	0.9	0.5	0.8
y4	0.6	0.7	0.5

Then their composition is

$$T = R \circ S = \begin{pmatrix} & z1 & z2 & z3 \\ x1 & 0.6 & 0.8 & 0.5 \\ x2 & 0.0 & 0.4 & 0.0 \\ x3 & 0.7 & 0.9 & 0.7 \end{pmatrix}$$

What is the physical interpretation of the relation T?

7. Algebraic Product of Fuzzy Relations

Let R and S be fuzzy relation in VxW. The algebraic product of the two fuzzy relations R and S in the space VxW is defined as a fuzzy set in VxW, whose elements satisfy the relation,

AlgProduct(R, S)(v, w) = R(v, w) * S(v, w). For all (v, w) in VxW,

Example:

R = [{{{1,1},0.8},{{1,2},0.3},{{1,3},0.5},{{1,4},0.2},
{{2,1},0.4},{{2,2},0},{{2,3},0.7},{{2,4},0.3},
{{3,1},0.6},{{3,2},0.2},{{3,3},0.8},{{3,4},0.6}}]

S = [{{{1,1},0.9},{{1,2},0.5},{{1,3},0.8},{{1,4},1},
{{2,1},0.4},{{2,2},0.6},{{2,3},0.7},{{2,4},0.5},
{{3,1},0.7},{{3,2},0.8},{{3,3},0.8},{{3,4},0.7}}]

So,
(R **U**S) = [{{{1,1},0.72},{{1,2},0.15},{{1,3},0.40},{{1,4},0.2},
{{2,1},0.16},{{2,2},0},{{2,3},0.49},{{2,4},0.15},
{{3,1},0.42},{{3,2},0.16},{{3,3},0.16},{{3,4},0.42}}]

8. Max-Star Composition

The max-star composition is defined such that for all (u, w) in UxW,

MaxStar (R1, R2) (u, w) = Max (R1 (u, v)*R2 (v, w)) over all v in the set V.

If we take the algebraic product for the star operation, then we will obtain a composition referred to as the max-product, which is defined in the following way:

For all (u, w) in UxW,

MaxProduct(R1, R2)(u, w) = Max(R1(u, v) x R2(v, w)) over all v in the set V.

Example:

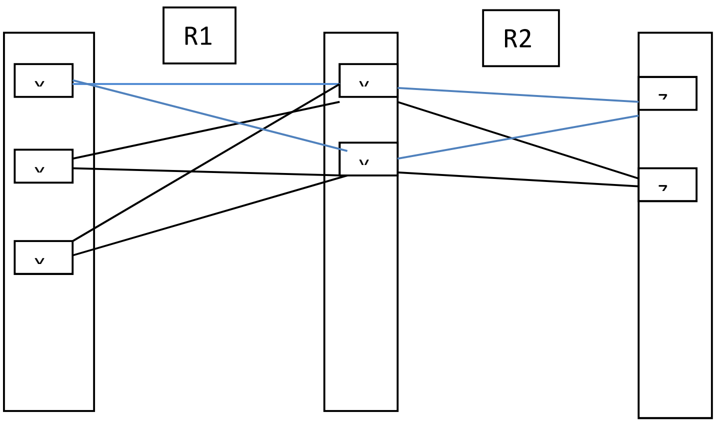
R = [{{{1,1},0.1},{{1,2},0.2},{{1,3},0},
{{2,1},1},{{2,2},0.7},{{2,3},0.3},
{{3,1},0.5},{{3,2},0},{{3,3},0.2},
{{4,1},1},{{4,2},0.8},{{4,3},0},
{{5,1},1},{{5,2},0.4},{{5,3},0.3}}]

S = [{{{1,1},0.4},{{1,2},0.7},{{1,3},0.3},{{1,4},0.2},
{{2,1},0.3},{{2,2},1},{{2,3},0},{{2,4},0.1},
{{3,1},0.8},{{3,2},0.4},{{3,3},0.5},{{3,4},0.6}}]

MaxMin (R1, R2) = [{{{1,1},0.06},{{1,2},0.2},{{1,3},0.03},{{1,4},0.02},
{{2,1},0.4},{{2,2},0.7},{{2,3},0.3},{{2,4},0.2},
{{3,1},0.2},{{3,2},0.35},{{3,3},0.15},{{3,4},0.12},
{{4,1},0.4},{{4,2},0.8},{{4,3},0.3},{{4,4},0.2},
{{5,1},0.4},{{5,2},0.7},{{5,3},0.3},{{5, 4},0.2}}]

INTUITIVE REPRESENTATION OF COMPOSITION OPERATIONS

Consider the relations R1 and R2 shown above as shown in the figure. Then, the composition operation consists of findings the maximum of the strength of the two paths, where the strength of each individual path is the minimum (or the product). An example of the paths for (x1, z1) is highlighted in the Figure 6.



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